

Optimal and Secure Measurement Protocols for Quantum Sensor Networks

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Studies of quantum metrology have shown that the use of many-body entangled states can lead to an enhancement in sensitivity when compared to product states. In this paper, we quantify the metrological advantage of entanglement in a setting where the quantity to be measured is a linear function of parameters coupled to each qubit individually. We first generalize the Heisenberg limit to the measurement of non-local observables in a quantum network, deriving a bound based on the multi-parameter quantum Fisher information. We then propose a protocol that can make use of GHZ states or spin-squeezed states, and show that in the case of GHZ states the procedure is optimal, i.e., it saturates our bound.

Entanglement is a valuable resource in precision measurement, as measurements using entangled probe systems have fundamentally higher optimal sensitivity [1] than those using product states. A generic measurement using N unentangled probes will have a standard deviation from the true value asymptotically proportional to $1/\sqrt{N}$, while a measurement making full use of entanglement can improve this scaling to $1/N$. For measurement of a single parameter coupled independently to each probe system, this $1/N$ (or “Heisenberg”) scaling is the best possible that is consistent with the Heisenberg uncertainty principle [1, 2]. The procedure can also be reversed—enhanced sensitivity to disturbances can provide experimental evidence of entanglement [3–5].

Measurements making use of entanglement usually couple one parameter to N different systems [1, 6, 7]. However, the emerging potential of long-range quantum information opens new avenues for metrology [8, 9] and entanglement distribution [10]. The ability to distribute entanglement across spatially separated regions has already been used for recent loophole-free tests of Bell’s inequality [11–13]. In this work, we are interested in coupling N parameters to N different systems, which may be spatially separated, and measuring a linear function of all of them (see Fig. 1a), such as a single mode of a spatially varying field. Such measurements may be of interest in geodesy or geophysics [14–17].

The function q we wish to measure is a weighted sum of the individual parameters θ_i , where i indexes the individual systems and each weight is denoted by a known real number α_i ,

$$q = \sum_{i=1}^N \alpha_i \theta_i = \vec{\alpha} \cdot \vec{\theta}. \quad (1)$$

In this paper, we characterize the advantage entanglement provides in this setting and construct an optimal strategy that amounts to turning some qubits’ evolution “on” and “off” for time proportional to the weight with

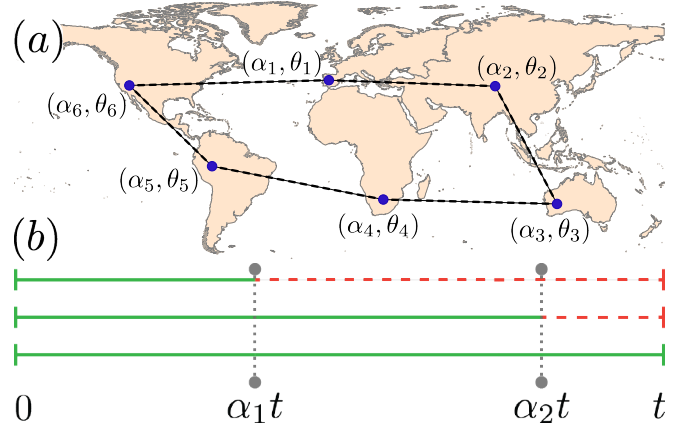


FIG. 1. (a) An illustration of the network setup. Stations, which are distant from each other, share an entangled state; at each station there is both an unknown parameter θ_i and a known relative weight α_i . We are concerned with estimating $\vec{\alpha} \cdot \vec{\theta}$. (b) Illustration of the partial time evolution protocol for three qubits. Solid green segments of the timeline represent periods when a qubit is evolving due to coupling to the local parameter θ_i , while dashed red segments represent periods after the qubit stops evolving. The switch occurs at times corresponding to their weight in the final linear combination. The weight of the last qubit is $\alpha_3 = 1$.

which their parameter contributes to the function q (see Fig. 1b). With this scheme of “partial time evolution,” we can measure a linear function with the minimum variance permitted by quantum mechanics, which can be viewed as an extension of the Heisenberg limit to multi-parameter problems. We will also show that our method can protect the secrecy of the result, allowing the network as a whole to perform a measurement without eavesdroppers learning any details of $\vec{\alpha} \cdot \vec{\theta}$.

It may seem surprising that the optimal measurement is one in which most qubits spend some of the measurement time idle. Since more time yields more signal, intuition suggests that the most effective strategy would

make better measurements on the less-weighted qubits rather than keep them off for much of the measurement time. This reasoning fails because we assume absolutely no prior knowledge on the individual parameters. Consider attempting to measure $\theta_1 + \theta_2$. If θ_1 is known to high precision, but we lack any information about θ_2 , there is no way to use our certainty about θ_1 to improve an estimate of $\theta_1 + \theta_2$. Because we restrict ourselves to a state of ignorance about the individual parameters, only a measurement of the entire function is usable, and in this case our scheme is optimal.

Setup.—We consider a system in which there are N sensor stations. Each sensor station i possesses a single qubit coupled to an external parameter θ_i unique to each station. We suppose that the state as a whole evolves unitarily under the Hamiltonian

$$\hat{H} = \tilde{H}(t) + \sum_{i=1}^N \frac{1}{2} \theta_i \hat{\sigma}_i^z. \quad (2)$$

Here, $\tilde{H}(t)$ is a time-dependent control Hamiltonian chosen by us which may include coupling to additional ancilla qubits. We wish to measure a quantity q defined in Eq. (1). We assume that $\forall i : |\alpha_i| \leq 1$, and additionally that there is at least one α_i such that $\alpha_i = 1$. These conditions simply set a scale for the function, and for an arbitrary $\vec{\alpha}$ all that is needed is division by the largest α_i to meet this requirement. As an example, a network with two stations interested in measuring the contrast between those stations would set $\vec{\alpha} = (1, -1)$ to measure $\theta_1 - \theta_2$. We would like to establish how well an arbitrary measurement of $\vec{\alpha} \cdot \vec{\theta}$ can be made, and what the best measurement protocol is for doing so. By “protocol” we mean three different choices: (1) which input state we begin with, (2) what auxiliary control Hamiltonian $\tilde{H}(t)$ we implement, and (3) how the final measurement is to be made.

We define the quality of measurement in terms of an estimator, Q , constructed from experimental data. (Throughout this paper, we denote operators with hats, quantities by lowercase, and corresponding estimators by uppercase.) We assume that the estimator is unbiased, so that its average value is the true value $\mathbf{E}[Q] = q$. Then our metric for the quality of the measurement is the average squared error, or variance, of the estimator,

$$\text{Var } Q = \mathbf{E}[(Q - q)^2]. \quad (3)$$

If measurements of θ_i can be made locally with accuracy $\text{Var } \Theta_i$, then we could compute the linear combination by local measurements and classical computation. In this case, the variance is given by classical statistical theory as $\text{Var } Q = \|\vec{\alpha}\|^2 \text{Var } \Theta_0$, assuming that $\text{Var } \Theta_i$ is identical at each site and equal to $\text{Var } \Theta_0$. A measurement of an individual frequency can be made with a variance of $1/t^2$

[2]. Therefore, our entanglement-free figure of merit is

$$\text{Var } Q \geq \frac{\|\vec{\alpha}\|^2}{t^2}. \quad (4)$$

We consider this the standard quantum limit (SQL) for networks. To compare to the typical case where N independent qubits measure a single parameter, consider the average $\bar{\theta}$, which is equivalent to setting all $\alpha_i = 1$ and then using $\bar{\Theta} = Q/N$ to obtain $\text{Var } \bar{\Theta} = 1/Nt^2$. It is our goal in this paper to present a means to improve on this figure using entanglement.

Optimality.—Our task is to perform parameter estimation on a quantum system evolving under some set of parameters $\{\theta_i\}$ linearly coupled to sensor qubits as in Eq. (2) [18–21]. Note that even though we are only interested in measuring a single number, we still need to treat a system which has many parameters in the evolution, necessitating the use of a full multi-parameter theory as in Refs. [22–29]. It is known from classical estimation theory that, given a probability distribution $p(z)$ over a set of outcomes z that depends on a number of parameters, all estimators of the parameters obey the Cramér-Rao inequality [30, 31],

$$\Sigma \geq \frac{F^{-1}}{M}. \quad (5)$$

Here, M is the number of experiments performed, F is the Fisher information matrix (see below), and Σ is the covariance matrix, where $\Sigma_{ij} = \mathbf{E}[(\Theta_i - \theta_i)(\Theta_j - \theta_j)]$. The inequality is a matrix inequality, meaning that $M\Sigma - F^{-1}$ is positive semidefinite. We will concern ourselves with the single-shot Fisher information, and set $M = 1$ from now on. The Fisher information matrix captures how each parameter changes the probability distribution of outcomes,

$$F_{ij} = \int p(z) \left(\frac{\partial \ln p_z}{\partial \theta_i} \right) \left(\frac{\partial \ln p_z}{\partial \theta_j} \right) dz. \quad (6)$$

This bound is a purely classical statement about probability distributions, and is saturated asymptotically using a maximum-likelihood estimator [32]. Quantum theory bounds the probability distributions which can result from a state evolved under a parameter-dependent unitary operation.

Our parameters in the Fisher information matrix F are the individual θ_i , while the quantity we want to estimate is some combination of these, $\vec{\alpha} \cdot \vec{\theta}$. To formulate the appropriate Cramér-Rao bound in this case, we write a new Fisher information matrix, G , where the parameters are recast in a new basis which includes $\vec{\alpha}$,

$$G_{ij} = \int p_z \left(\frac{\partial \ln p_z}{\partial (\vec{\alpha}^{(i)} \cdot \vec{\theta})} \right) \left(\frac{\partial \ln p_z}{\partial (\vec{\alpha}^{(j)} \cdot \vec{\theta})} \right) dz. \quad (7)$$

Here, $\vec{\alpha}^{(i)}$ represents an element of a new set of N real vectors which span the entire N -dimensional space. We designate the first element of that set, $\vec{\alpha}^{(1)}$, as the $\vec{\alpha}$ we are interested in. The Cramér-Rao bound on our quantity of interest is

$$\text{Var } Q \geq (G^{-1})_{11}. \quad (8)$$

Using the chain rule, it can be shown that if we arrange the vectors $\vec{\alpha}^{(i)}$ into a matrix A such that $A_{ij} = \alpha_j^{(i)}$, F is related to G by the transformation $F = A^T G A$. It then follows that $(G^{-1})_{11} = \vec{\alpha}^{(1)T} F^{-1} \vec{\alpha}^{(1)} = \vec{\alpha}^T F^{-1} \vec{\alpha}$, so that

$$\text{Var } Q \geq \vec{\alpha}^T F^{-1} \vec{\alpha}. \quad (9)$$

Since F is positive semidefinite, \sqrt{F} is Hermitian. We can then write the following for an arbitrary real \vec{b} by invoking the Cauchy-Schwarz inequality:

$$\|\sqrt{F^{-1}}\vec{\alpha}\|^2 \|\sqrt{F}\vec{b}\|^2 \geq \|\vec{\alpha}^T \sqrt{F^{-1}} \sqrt{F}\vec{b}\|^2. \quad (10)$$

Taking \vec{b} to be the b th element of the standard basis gives

$$\text{Var } Q \geq \vec{\alpha}^T F^{-1} \vec{\alpha} \geq \frac{\alpha_b^2}{F_{bb}}. \quad (11)$$

Since F_{bb} is the Fisher information for a single parameter, for any $\tilde{H}(t)$ including ancilla qubits, $F_{bb} \leq t^2 \|\hat{h}_b\|_s^2$ [20], where $\|\hat{h}_b\|_s$ is the operator seminorm of the generator corresponding to parameter θ_b . The seminorm is the difference between the largest and smallest eigenvalues of \hat{h}_b . Our final bound comes from applying this condition and recognizing that the formula must hold for all b :

$$\text{Var } Q \geq \max_b \frac{\alpha_b^2}{t^2 \|\hat{h}_b\|_s^2}. \quad (12)$$

In Eq. (2), all $h_b = \frac{1}{2} \hat{\sigma}_b^z$, $\|\hat{h}_b\|_s = 1$, and we find a bound,

$$\text{Var } Q \geq \frac{1}{t^2}. \quad (13)$$

Here we have also used the fact that the largest $\alpha_i = 1$. If we want to calculate an average, then all qubits are equally weighted and the desired quantity is $\bar{\theta} = q/N$, so $\text{Var } \bar{\theta} \geq 1/N^2 t^2$ has the desired Heisenberg scaling. However, note that if we wanted to calculate only a single θ_i , then we would not scale coefficients at all. In this case, there is obviously no advantage from having more qubits. Our bound allows us to explore the full range of possible $\vec{\alpha}$ between these two extremes.

We can verify that the above argument is correct by optimizing over the space of all control Hamiltonians $\tilde{H}(t)$. As this is computationally expensive, we limit ourselves to a two-qubit sensor network with no ancillas. The parameters we optimize over include enough operators to

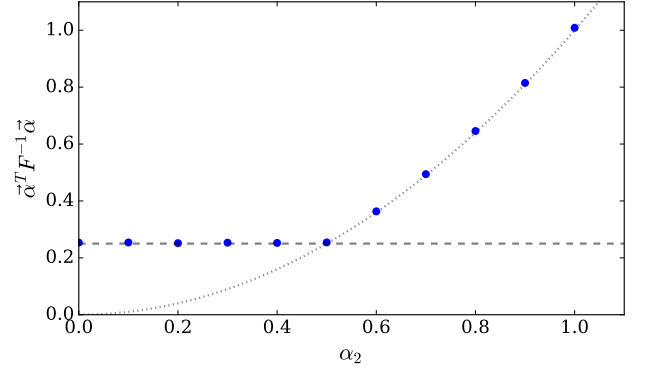


FIG. 2. Numerical optimization of $\vec{\alpha}^T F^{-1} \vec{\alpha}$ for two qubits with $\alpha_1 = 1$ compared to the bound predicted by our analytic result. Each point is generated by 200 rounds of optimization using a gradient descent algorithm; the control parameters begin at small random values. The dashed (dotted) line is the analytic bound derived from the first (second) qubit. As α_2 increases, the second qubit becomes the source of the relevant bound.

provide universal control on two qubits, meaning we can effectively modify the input state as well as the final measurement basis in order to optimize the Fisher information. In order to test the form of our bound Eq. (12), which depends both on relative weights of each parameter and the underlying generator, we couple θ_1 to a generator $\hat{\sigma}_1^z$, which has $\|\hat{\sigma}_1^z\|_s = 2$, while leaving the second qubit coupled to a generator $\frac{1}{2} \hat{\sigma}_2^z$ as in Eq. (2). The bound corresponding to the first qubit from Eq. (12) is $\alpha_1^2/4t^2$, and that of the second qubit is α_2^2/t^2 . In our numerics, we set $\alpha_1 = t = 1$, meaning the two bounds are $1/4$ and α_2^2 . Our analytic result leads us to believe therefore that if $\alpha_2^2 > 1/4$, the minimum possible variance should be α_2^2 . However, if $\alpha_2^2 < 1/4$, then the lower bound should be $1/4$. That behavior is precisely what we find through brute force numerical optimization, as shown in Fig. 2, confirming Eq. (12).

Protocol.—We now present a protocol that saturates the bound of Eq. (13). We start by considering an N -qubit Greenberger-Horne-Zeilinger (GHZ) state:

$$\frac{1}{\sqrt{2}} (|0\rangle^{\otimes N} + |1\rangle^{\otimes N}). \quad (14)$$

Under $\hat{\sigma}^z$ evolution, each $|1\rangle$ accumulates a phase relative to $|0\rangle$. By allowing qubits to accumulate phase proportional to the desired weight α_i , we can get a final state in which $|1\rangle^{\otimes N}$ has accumulated a total phase of $\vec{\alpha} \cdot \vec{\theta} t$ relative to $|0\rangle^{\otimes N}$. We refer to our protocol as “partial time evolution” because it relies on a qubit undergoing evolution for a fraction of the total measurement time. We can realize this by applying $\hat{\sigma}_i^x$ to a qubit at time $t_i = t(1 + \alpha_i)/2$ so that the qubit evolution will be identical to evolving it for a time $\alpha_i t$. Note that if there is a

fixed experimental time t , this scheme can realize values of $\alpha_i \in [-1, 1]$, which motivates our restrictions on the values of individual α_i .

The result of this protocol is an effective evolution according to the unitary operator

$$U(t) = e^{-i\frac{t}{2} \sum_{i=1}^N \alpha_i \theta_i \hat{\sigma}_i^z}. \quad (15)$$

Under this evolution, the final state is:

$$\frac{1}{\sqrt{2}} \left(e^{i\frac{t}{2}q} |0\rangle^{\otimes N} + e^{-i\frac{t}{2}q} |1\rangle^{\otimes N} \right). \quad (16)$$

Now we make a measurement of the overall parity of the state, $\hat{P} = \bigotimes_{i=1}^N \hat{\sigma}_i^x$. The details of this measurement and calculation of $\langle \hat{P} \rangle$ are given in Ref. [1]. Measurement of the time-dependent expectation value $\langle \hat{P} \rangle(t)$ allows for the estimation of Q with accuracy [33]

$$\text{Var } Q = \frac{\text{Var } \hat{P}(t)}{(\partial \langle \hat{P} \rangle / \partial t)^2} = \frac{\sin^2 qt}{t^2 \sin^2 qt} = \frac{1}{t^2}, \quad (17)$$

saturating the bound in Eq. (13).

One advantage of this protocol is that an eavesdropper cannot learn the result of the network measurement by capturing a subset of the nodes' measurement results. This privacy can be shown by tracing out the first qubit in Eq. (16), which leaves no phase information in the resulting mixed state. The central node can receive the measurement outcomes from all other nodes but keep its own secret, and no eavesdropper is able to extract information from the broadcasted results, even if the central node's qubit is unweighted (i.e., $\alpha_i = 0$).

Input States and Decoherence.—The perfect security of the GHZ state stems from the fact that a single lost qubit ruins the measurement, but this also manifests as extreme sensitivity to noise which can pose problems for metrology [34–36]. For this reason, it can be advantageous to use other entangled states that may be more robust to decoherence, although these will be generally less secure as well.

One option is to use spin-squeezed states, which have known advantages for precision measurement. These are collective spin states which, due to entanglement, have reduced variance of angular momentum along one axis at the cost of increased variance along an orthogonal axis [2, 37]. Recently, it has been shown that these states may allow Heisenberg scaling measurements even without single-particle detection, which makes them very attractive for experimental implementations [38]. In particular, we consider a state whose overall spin vector is aligned along $+x$, such that $\langle \hat{\sigma}_i^x \rangle \approx 1$. We assume that the other components of individual spin have zero expectation value, but that the variance of the collective spin projection $\hat{J}_y = \frac{1}{2} \sum \hat{\sigma}_i^y$ is decreased while the variance of \hat{J}_z is increased. We quantify this effect through the

spin-squeezing parameter ξ [2],

$$\xi = \sqrt{\frac{\text{Var } \hat{J}_y}{N/4}}. \quad (18)$$

Suppose that we perform Ramsey interferometry on such a state [2, 33]. The protocol includes both partial time evolution $U(t)$ and a final rotation pulse $R_x(\frac{\pi}{2}) = \exp(-i\frac{\pi}{4} \sum \hat{\sigma}_i^x)$. The final measurement will be of the total angular momentum projection J_z after applying these operations:

$$\langle \hat{J}_z(t) \rangle = \langle U^\dagger(t) R_x^\dagger\left(\frac{\pi}{2}\right) \hat{J}_z(0) R_x\left(\frac{\pi}{2}\right) U(t) \rangle, \quad (19)$$

$$= \frac{1}{2} \left\langle \sum_{i=1}^N \hat{\sigma}_i^x \sin \alpha_i \theta_i t + \hat{\sigma}_i^y \cos \alpha_i \theta_i t \right\rangle. \quad (20)$$

If we specify that this expectation is to be taken over a squeezed state with $\langle \hat{\sigma}_i^x \rangle \approx 1$ and $\langle \hat{\sigma}_i^y \rangle = 0$, then our signal will be sensitive only to $\vec{\alpha} \cdot \vec{\theta}$ if each individual phase is small:

$$\langle \hat{J}_z(t) \rangle_{\text{squeezed}} = \frac{1}{2} \sum_{i=1}^N \sin \alpha_i \theta_i t \approx \frac{t}{2} \sum_{i=1}^N \alpha_i \theta_i. \quad (21)$$

This shows that a squeezed state can be used for measurement of heterogeneous parameters. The sensitivity can then be calculated just as before,

$$\text{Var } Q = \frac{\text{Var } \hat{J}_z(t)}{(\partial \langle \hat{J}_z(t) \rangle / \partial q)^2} \bigg|_{q=0} = \frac{\text{Var } \hat{J}_y}{t^2/4} = \frac{N\xi^2}{t^2}. \quad (22)$$

We evaluate the sensitivity at $q = 0$ because we are interested in small signals. Partial time evolution with spin-squeezed input beats the SQL if $\xi \leq \|\vec{\alpha}\|/\sqrt{N}$. Squeezed states can achieve squeezing proportional to $N^{-1/2}$ [2, 37], which approaches our bound in Eq. (13) up to numerical prefactors.

Other highly-entangled states such as Dicke states also have metrological value in the presence of noise and could also serve as input states to partial time evolution with similarly favorable scaling [29, 39–42]. Besides using other input states, techniques like dynamical decoupling [43] or quantum error correction could be explored in efforts to mitigate decoherence [44, 45].

Outlook.—We have presented a measurement protocol for quantum networks which is useful for measuring linear combinations of parameters. Measurement schemes which use many-body entangled states have been proposed for clocks [7, 34, 46] and may also prove useful for atomic magnetometry [47], gravimetry [48, 49], spectroscopy [6], and rotation sensing [50–52]. While we focused on sensors distributed over long distances, there may also be applications at small scales. Recent experiments have used atomic magnetometers to detect animal neurons firing [53] or perform MRI [54]. Studying

the spatial modes of these phenomena through the techniques presented here could prove valuable.

In future work, we also would like to quantify the power of a measurement in a Bayesian setting where a prior on individual θ_i exists, which will require more statistical sophistication. We hope to understand whether and how the existence of prior distributions opens up new measurement strategies beyond those presented here [55–57].

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